DEUTSCHES ELEKTRONEN – SYNCHROTRON

DESY 91-071
June 1991

受入
高工研図書室

Theory of Multi-Bunch Feedback Systems

R.D. Kohaupt

Deutsches Elektronen-Synchrotron DESY, Hamburg

ISSN 0418-9833

NOTKESTRASSE 85 · D-2000 HAMBURG 52

DESY behält sich alle Rechte für den Fall der Schutzrechtserteilung und für die wirtschaftliche Verwertung der in diesem Bericht enthaltenen Informationen vor.

DESY reserves all rights for commercial use of information included in this report, especially in case of filing application for or grant of patents.

To be sure that your preprints are promptly included in the
HIGH ENERGY PHYSICS INDEX,
send them to the following (if possible by air mail):

DESY
Bibliothek
Notkestrasse 85
D–2000 Hamburg 52
Germany

DESY 91-071
June 1991

ISSN 0418-9833

Theory of Multi-Bunch Feedback Systems

R.D. Kohaupt

Deutsches Elektronen-Synchrotron DESY, Hamburg

Abstract

In this article the theory of multibunch feedback systems is developed in a rigorous way including the fact that the elements of feedback systems are localized in the ring. The results of the theory which can be used for any strength of the systems are the base for the multibunch feedback systems for PETRA and HERA, already tested successfully in PETRA.

1. Introduction

This article presents the basic theoretical tool for the longitudinal and transverse multi-bunch feedback systems which have been installed and successfully tested in PETRA-e^- [1, 2]. Slightly modified systems are also being built for HERA-e^-. Since the threshold currents at the natural damping rates are about a factor 20 below the designed operational currents for both machines, the feedback systems have to provide a strong damping over the total bandwidth of 5 MHz in order to cure all unstable coupled bunch motions induced by the parasitic modes of the accelerating cavities.

The damping of a harmonic oscillator seems to be a "trivial" problem and usually the design of damper systems for beam motion is also based on damped oscillators.

However, the realization of a strong damper system for multi-bunch motion is very different from the naive oscillator model.

First of all, the pick-up system and the devices acting on the beam are localized objects, which is an essential feature in the case of strong damping.
Since the data detection at a local station, the signal processing and the transfer to the acting devices all take time which usually exceeds one turn of revolution, the damper systems have to operate with a several turn delay.
Although the localization of the feedback elements offers the method of transfer matrices, a variable number of delayed turns cannot be described in terms of transfer matrices with a fixed size.
In the following chapters, the theory of multi-bunch dipole feedback systems will therefore be developed in the framework of linear differential equations with strongly time dependent driving terms.

2. Basic Considerations

Let us start with a simple oscillator of the following form:

$$\ddot{y}(t) + \omega^2 y(t) = F[y(t)] + g(t) \tag{1}$$

Here $F[y(t)]$ is a linear functional describing a "feedback" term and $g(t)$ is an external excitation. On eq. (1) we impose the conditions

$$y(t) \equiv 0 \quad \text{for } t \leq 0 \tag{2}$$
$$g(t) \equiv 0 \quad \text{for } t \leq 0$$

so that the system is completely at rest for $t \leq 0$. For $t \geq 0$ the function $g(t)$ serves as a short "test" excitation, which then shows whether the system (1) is stable or not.

In order to solve (1) we introduce the generalized Fourier transform with a complex "frequency" w (3), (4)

$$F(y) = \tilde{y}(w) = \frac{1}{2\pi}\int_0^\infty dt\, y(t) e^{-iwt} \tag{3}$$

$$\text{for } Im\,w < -\alpha$$

where α is a positive real constant such that (3) will exist. The inverse transform then reads

$$y(t) = \int_K dw\ \tilde{y}\,(w)e^{iwt} \tag{4}$$

where the integration has to be performed along a path K in the lower complex plane so that $Imw < -\alpha$. The functional F is finally defined as

$$F[y(t)] = \int dt' F(t-t')y(t'). \tag{5}$$

Because of causality $F(t-t')$ obeys

$$F(t-t') \equiv 0 \ \text{ for } t < t' \tag{6}$$

The generalized Fourier transform for F reads

$$\tilde{F}\,(w) = \int d\zeta F(\tau)e^{-iw\zeta} \tag{7}$$

and correspondingly

$$F(\zeta) = \frac{1}{2\pi}\int_K dw\ \tilde{F}\,(w)e^{iw\zeta}\ \ ;\zeta = t-t' \tag{8}$$

Finally we find for eqs. (3),(4):

$$\begin{aligned} F(y) &= \tilde{y}\,(w) \\ F(\dot{y}) &= iw\ \tilde{y}\,(w) - y(0) \\ F(\ddot{y}) &= w^2\,\tilde{y}\,(w) - iwy(0) - \dot{y}\,(0). \end{aligned} \tag{9}$$

Because of the "start point" condition (2) we can drop the $y(0)$ terms on the r.h.s. of (9). The functions $\tilde{y}\,(w)$, $\tilde{F}\,(w)$, $\tilde{g}\,(w)$ are analytic functions in the lower complex plane, at least for

$$Imw < -\alpha$$

where α is determined by the stability properties of eq. (1).

Performing the generalizing Fourier transform of (1), we obtain according to (3), (5) and (9)

$$\tilde{y}\,(w) = \frac{\tilde{g}\,(w)}{\omega^2 - w^2 - \tilde{F}\,(w)} \tag{10}$$

Since the Fourier transforms are analytic functions in the lower complex plane the time behaviour $y(t)$ is completely determined by the analytic properties of $\tilde{y}\,(w)$ in the plane $Imw > -\alpha$.
For an exciting kick $g(t) \sim \delta(t)\ \tilde{g}\,(w)$ is a constant, so that $y(t)$ is determined by the poles of $\tilde{y}\,(w)$ - or equivalently - by the zeroes of the denominator

$$\omega^2 - w^2 = \tilde{F}\,(w). \tag{11}$$

In order to provide damping, $\tilde{F}\,(w)$ has to be arranged such that the solutions of (11) are of the form

$$w_s = \omega_s + i\delta_s \ \text{ with } \delta_s > 0. \tag{12}$$

The transfer function $\tilde{F}\,(w)$ can be separated with respect to amplitude and phase:

$$\tilde{F}\,(w) = |\,\tilde{F}\,(w)|e^{i\chi(w)}. \tag{13}$$

As an example let us consider a pure delay:

$$\tilde{F}\,(w) = |\,\tilde{F}\,(w)|e^{-i\Delta t w}. \tag{14}$$

Then (11) becomes

$$\omega^2 - w^2 = |\,\tilde{F}\,(w)|e^{-i\Delta t w}. \tag{15}$$

This equation has an infinite number of solutions δ_n, ω_n where $\delta_n, \omega_n \longrightarrow \infty$ for $n \longrightarrow \infty$. However, an arbitrarily high ω_n together with a high damping rate is not a reasonable separation for strong focussing in an accelerator, so that (15) resp. (11) fails in that case. Usually in the case of weak forces, i.e. in the case of small $|\,\tilde{F}\,(w)|$ one considers the approximation of (11) according to

$$(\omega - w) \approx \frac{1}{2\omega}|\,\tilde{F}\,(\omega)|e^{-i\omega\Delta t}. \tag{16}$$

This can then serve only as an estimate of the damping rate (and frequency shift).
Therefore (1) has to be replaced by an equation containing the localized driving terms, leading to a modification of (11) which is valid for any feedback strength and any delay.

3. Filtering of Sampled Data

At a local pick-up station the deviations from the equilibrium beam position are given in terms of values $y(kT)$ sampled after the k-th turn, T being the revolution time. Let us consider the time dependent signal $S(t)$ defined by

$$S(t) = \sum_{(k)} X(kT)T\delta(t-kT) \ \text{ with } \sum_{(k)} = \sum_{k=-\infty}^{+\infty} \tag{17}$$

which is equivalent to

$$S(t) = \sum_{(k)} X(t)T\delta(t-kT). \tag{18}$$

Applying a filtering G on $S(t)$ yields using (5) and (7)

$$G[S(t)] = \sum_{\nu}\int dw\ \tilde{G}\,(w)e^{iwt}\,Te^{-iw\nu T}X(\nu T). \tag{19}$$

The time dependent signals delivered by the detection system do not, of course, contain Dirac-pulses but the output of those systems can be described by signals of the form

$$\bar{S}(t) = \sum_{(k)} y(kT)T\Delta(t, kT) \tag{20}$$

with

$$\Delta(t,kT) = \frac{1}{\Delta t}\Theta(t-kT)\Theta(kT+\Delta t - t). \tag{21}$$

In order to evaluate $G[\overline{S}(t)]$ we first evaluate

$$\tilde{\Delta}_k(w) = \frac{1}{2\pi}\frac{1}{\Delta t}\int_{kT}^{kT+\Delta t} e^{-iwt'}dt' = \frac{1}{2\pi}e^{-iw\Delta t/2}\frac{\sin w\Delta t/2}{w\Delta t/2}e^{-ikTw} \tag{22}$$

which yields

$$\tilde{\Delta}_k(w) = e^{-ikTw}\,\tilde{\Delta}(w). \tag{23}$$

After this we obtain

$$G[\overline{S}(t)] = \sum_{(k)}\int_K dw\,\tilde{G}(w)\,\tilde{\Delta}(w)Te^{iwt}e^{-iwkT}X(kT), \tag{24}$$

which can be related to $S(t)$ by

$$G[\overline{S}(t)] = \overline{G}[S(t)]$$

with

$$\tilde{\overline{G}}(w) = \tilde{G}(w)\cdot\tilde{\Delta}(w), \tag{25}$$

so that the "realistic" signals can also be related to Dirac pulses using a modified transfer function. In the following sections we will therefore rely on signals like (17).

4. The Multi-Bunch Feedback System

A. longitudinal direction

Signals

We consider a system of N bunches equally spaced around the ring with the same intensity. The bunches are numbered by $\nu = 0,\ldots,N-1$. The phase deviations of the bunches at a time t are defined as $\Phi_\nu(t-\nu T_B)$ where $T_B = T/N$. The voltage applied to the beam through the feedback cavities is then described by a transfer of the time dependent signal

$$S(t) = a\sum_\nu\sum_{(k)}\Phi_\nu(t-\nu T_B)\delta(t-kT-\nu T_B)T \tag{26}$$

according to

$$\Delta U(t) = bG[S(t)] \tag{27}$$

where a, b are "strength" factors.

Equations of motions

The equations of motion for longitudinal dipole oscillations read

$$\frac{d^2}{dt^2}\Phi_\mu(t-\mu T_B) + \Omega_S^2\Phi_\mu(t-\mu T_B) = \frac{\Omega_S^2}{U_c}\sum_{(k)}\Delta U(kT+\mu T_B)T\delta(t-kT-\mu T_B) + \tag{28}$$

$$+\sum_{(k)}\varphi(t)T\delta(t-kT-\mu T_B),\ \mu = 0,\ldots,N-1.$$

Here Ω_S denotes the circular synchrotron frequency and U_c the accelerating peak voltage of the main rf system, $\varphi(t)$ describes an external excitation.
It has been assumed that the time of action at each turn is extremely short compared to a synchrotron period so that the forces can be described by δ-functions. This is not a necessary restriction but avoids a tremendous blow-up of the formalism.

Signal transfer

We introduce the generalized Fourier transform of Φ_ν and rewrite (26)

$$S(t) = aT\sum_\nu\sum_{(k)}\int_L dw'\,\tilde{\Phi}_\nu(w')e^{-iw'\nu T_B}e^{ikw_0t}e^{-ikw_0\nu T_B}e^{iw't} \tag{29}$$

with $\omega_0 = 2\pi/T$. The integration is performed along a path L with $Imw' = \overline{\alpha} < -\alpha$. Expanding the $\tilde{\Phi}_\nu(w')e^{-iw'\nu T_B}$ in terms of "normal modes" according to

$$\tilde{\Phi}_\nu(w')e^{-iw'\nu T_B} = \sum_r\frac{1}{\sqrt{N}}\tilde{C}_r(w')e^{2\pi ir\nu/N};\ r = 0,\ldots,N-1 \tag{30}$$

together with

$$\sum_\nu e^{2\pi e(r-r')/N} = N\delta_{-lN,(r-r')} \tag{31}$$

for all integers l and $r, r' = 0,\ldots,N-1$ we obtain

$$S(t) = aT\sum_r\sum_{(l)}\int_L dw'\frac{1}{\sqrt{N}}e^{i(lN\omega_0+r\omega_0+w')t}\,\tilde{C}_r(w). \tag{32}$$

The transfer $f(t) = G[S(t)]$ through the total feedback system can be obtained from (32)

$$f(t) = aT\sum_r\sum_{(l)}\int_L dw'\frac{1}{\sqrt{N}}\tilde{C}_r(w')\,\tilde{G}(w'+r\omega_0+lN\omega_0)e^{i(lN\omega_0+r\omega_0+w')t}. \tag{33}$$

The voltage values of (28) follow from (32) for $t = kT+\mu T_B$.
Setting $aTb = g$ we write

$$\Delta U(KT+\mu T_B) = g\sum_r\int_L dw'\,\tilde{C}_r(w')\,\tilde{G}_N(w'+r\omega_0)e^{i(w'+r\omega_0)kT}e^{2\pi i\mu r/N}e^{iw'\mu T_B} \tag{34}$$

where we have introduced

$$\sum_{(l)}\tilde{G}(w'+r\omega_0+lN\omega_0) = \tilde{G}_N(w'+r\omega_0), \tag{35}$$

so that $\tilde{G}_N$ is a periodic function of w' with the period $N\omega_0$.

We introduce the functions

$$\Delta U_\mu(t) = \sum_{(k)}\Delta U(kT+\mu T_B)T\delta(t-kT-\mu T_B) \tag{36}$$

$$\varphi_\mu(t) = \sum_{(k)} \varphi(t) T \delta(t - kT - \mu T_B) \tag{37}$$

for convenience, we make the replacement $t - \mu T_B = t_\mu$ on the l.h.s. of equation (28), we arrive at

$$\frac{d^2}{dt_\mu^2}\Phi_\mu(t_\mu) + \Omega_s^2 \Phi_\mu(t_\mu) = \frac{\Omega_s^2}{U_C}\Delta U_\mu(t) + \varphi_\mu(t). \tag{38}$$

In order to find the generalized Fourier transform of (38) we have to derive this transform for $\Delta U_\mu(t)$ first

$$\Delta \tilde{U}_\mu(w) = \frac{1}{2\pi}\int dt_\mu \Delta_\mu U(t) e^{-iwt_\mu} \tag{39}$$

where for the present $Im w = \overline{\alpha}$.

$$\Delta\tilde{U}_\mu(w) = \frac{1}{2\pi}\int dt\, \Delta U_\mu(t) e^{-iw(t-\mu T_B)} = \frac{1}{2\pi} g \sum_{(k)} \Delta U(kT + \mu T_B) e^{-iwkT}\cdot T \tag{40}$$

Using (34) and

$$\frac{1}{2\pi}\sum_{(k)} e^{i(w' + r\omega_0 - w)kT} = f_0 \sum_{(k)} \delta[w' - (w - r\omega_0 + k\omega_0)]. \tag{41}$$

yields

$$\Delta\tilde{U}_\mu(w) = g\sum_r \sum_{(k)} \frac{1}{\sqrt{N}} C_r(w - r\omega_0 + k\omega_0)\, \tilde{G}_N(w + k\omega_0) e^{i(w+k\omega_0)\mu T_B}. \tag{42}$$

From the normal mode expansion (30) and the orthogonality properties of those modes we can derive

$$\frac{1}{\sqrt{N}}\sum_r \tilde{C}_r(w - r\omega_0 + k\omega_0) = \frac{1}{N}\sum_r\sum_\nu \tilde{\Phi}_\nu(w - r\omega_0 + k\omega_0) e^{-i(w+k\omega_0)\nu T_B}, \tag{43}$$

and finally

$$\Delta\tilde{U}_\mu(w) = g\sum_r\sum_{(k)}\sum_\nu \frac{1}{N}\tilde{\Phi}_\nu(w - r\omega_0 + k\omega_0)\, \tilde{G}_N(w + k\omega_0) e^{-i(w+k\omega_0)\nu T_B} e^{i(w+k\omega_0)\mu T_B} \tag{44}$$

It is shown in the Appendix A1 that this relation can be represented by

$$\Delta\tilde{U}_\mu(w) = g\sum_\nu \hat{\Phi}_\nu(w)\hat{G}_{\mu\nu}(w) \tag{45}$$

with

$$\hat{\Phi}_\nu(w) = \sum_{(l)} \tilde{\Phi}_\nu(w + l\omega_0) \tag{46}$$

and

$$\hat{G}_{\mu\nu}(w) = \sum_{(l)} \frac{e^{i(w+l\omega_0)\mu T_B}}{\sqrt{N}}\, \tilde{G}(w + l\omega_0) \frac{e^{-i(w+l\omega_0)\nu T_B}}{\sqrt{N}} \tag{47}$$

where $\hat{G}_{\mu\nu}$ and $\hat{\Phi}_\mu$ can now be analytically continued to arbitrary complex values of w according to the analytic properties of these functions. From (45)-(47) follows

$$\begin{aligned}\hat{\Phi}_\nu(w + n\omega_0) &= \hat{\Phi}_\nu(w)\\ \hat{G}_{\mu\nu}(w + n\omega_0) &= \hat{G}_{\mu\nu}(w)\end{aligned} \qquad \text{for all integers } n \tag{48}$$

Coupled-bunch equations

Using the equations (28), (38), and (45)-(47), the generalized Fourier transform of (28) can be finally derived

$$\tilde{\Phi}_\mu(w) = \frac{1}{\Omega_S^2 - w^2} g \frac{\Omega_S^2}{U_C}\sum_\nu \hat{G}_{\mu\nu}(w)\hat{\Phi}_\nu(w) + \frac{1}{\Omega_S^2 - w^2}\hat{\varphi}_\mu(w). \tag{49}$$

Since $\hat{\Phi}_\nu(w)$, $\hat{\varphi}_\mu(w)$ and $\hat{G}_{\mu\nu}(w)$ are periodic functions according to (45)-(47), the equation (49) can be replaced by

$$\hat{Q}(w)\cdot\hat{\Phi}_\mu(w) = g\frac{\Omega_S^2}{U_C}\sum_\nu \hat{G}_{\mu\nu}(w)\hat{\Phi}_\nu(w) + \hat{\varphi}_\mu(w) \tag{50}$$

where

$$\frac{1}{\hat{Q}(w)} = \hat{R}(w) = \sum_{(k)} \frac{1}{\Omega_S^2 - (w + k\omega_0)^2}. \tag{51}$$

The function $\hat{R}(w)$ has been evaluated in Appendix A2 and the result is

$$\hat{R}(w) = -\frac{i\pi}{\Omega_S\omega_0}\left\{\frac{1}{1 - ax} - \frac{1}{1 - a^* x}\right\}, \tag{52}$$

$$x = e^{-iwT};\quad a = e^{i\Omega_S T}. \tag{53}$$

For $w \longrightarrow \Omega$, we find from (52), (53)

$$\hat{Q}(w) \overset{w\to\Omega_s}{\longrightarrow} 2\Omega_S(\Omega_S - w). \tag{54}$$

The equations (50)-(51) are the basic equations for a feedback system with localized elements, valid for any strength and delay. The matrix $\hat{G}_{\mu\nu}(w)$ leads to a coupling of all the bunches, which is determined by the properties of the feedback transfer function.

Spectral representation of the coupling matrix

Before we look for solutions of (50)-(54), we first study the properties of $\hat{G}_{\mu\nu}(w)$. To this end we consider the state vectors $\vec{K}_r(w)$ with the νth component defined by

$$\overline{K}_{r\nu}(w) = \frac{1}{\sqrt{N}} e^{iw\nu T_B} e^{2\pi i r\nu/N}. \tag{55}$$

From (45)-(47) we obtain

$$\sum_\nu \hat{G}_{\mu\nu}(w)\overline{K}_{r\nu}(w) = \tilde{G}_N(w + r\omega_0)\overline{K}_{r\mu}(w) \tag{56}$$

so that each vector $\vec{\overline{K}}_T(w)$ is an eigenstate of the matrix $G_{\mu\nu}(w)$.

This means that $\hat{G}_{\mu\nu}(w)$ has a "spectral" representation of the form

$$\hat{G}_{\mu\nu}(w) = \sum_r \overline{K}_{r\mu}(w)\, \tilde{G}_N(w + r\omega_0)\overline{K}^*_{r\nu}(w) \tag{57}$$

with the complete set of orthonormal modes (modified "normal modes") defined by (55).

Eigenfrequencies of the system

These modes differ from the normal modes usually known by the factor $e^{i\omega\nu T_B}$ which takes into account the additional phase advance from bunch to bunch due to the tune of the oscillation. This factor is normally "neglected" in the naive approaches. Setting

$$\hat{\Phi}_\mu(w) = \sum_r \hat{C}_r(w)\overline{K}_{r\mu}(w),\ \hat{\varphi}_\mu(w) = \sum_r F_r(w)\overline{K}_{r\mu}(w) \tag{58}$$

the equations (50)-(54) become diagonal in the $\overline{K}_r$ base:

$$\hat{C}_r(w) = \frac{F_r(w)}{\hat{Q}(w) - g\frac{\Omega_S^2}{U_C}\tilde{G}_N\,(w + r\omega_0)}. \tag{59}$$

The stability behaviour will then be determined by the zeroes of the denominator

$$\hat{Q}(w) = g\frac{\Omega_S^2}{U_C}\,\tilde{G}_N\,(w + r\omega_0) \tag{60}$$

These are the basic equations for determining the eigenfrequencies of the system without any restriction on the feedback strength or on the number of turns in the case of signal delay. As a consequence, these equations also determine the limit for a reasonable gain and the maximum damping rates for a feedback system with localized elements.
Because of the reflection properties of the transfer function $\tilde{G}_N$

$$\tilde{G}_N^*\,(r\omega_0 + w) = G_N(-r\omega_0 - w^*) = \tilde{G}_N\,((N - r)\omega_0 - w^*) \tag{61}$$

and because of the properties

$$\overline{K}_{(N-r)\mu}(-w^*) = \overline{K}^*_{r\mu}(w) \tag{62}$$

together with

$$\hat{Q}(w) = \hat{Q}(r\omega_0 + w),\ \hat{Q}^*(w) = \hat{Q}(-w^*), \tag{63}$$

the complete set of eigenfrequencies guarantees real physical solutions.
If w_ϱ is a solution of (60), then also $w_\varrho + m\omega_0$ is a solution of (60) for $0 \le m \le N-1$. To prove this, we consider the system (60) for $r = \bar{\varrho}$ with

$$\bar{\varrho} = \varrho - m + \kappa(\varrho, m)\cdot N, \tag{64}$$

$$\kappa(\varrho, m) = \begin{cases} 0 & \text{for } m \le \varrho \\ 1 & \text{for } m > \varrho \end{cases} \tag{65}$$

and obtain

$$\hat{Q}(w_\varrho + m\omega_0) = \hat{Q}(w_\varrho) = g\frac{\Omega_S^2}{U_C}\,\tilde{G}_N\,(w_\varrho + (\bar{\varrho} + m)\omega_0) = g\frac{\Omega_S^2}{U_C}\,\tilde{G}_N\,(w_\varrho + \varrho\omega_0). \tag{66}$$

Since any integer n can be represented by $n = q\cdot N + m$ with $0 \le m \le N-1$ and an appropriate integer q we arrive at the following statement:

if w_ϱ is a solution of (60), then also $w_\varrho + l\omega_0$ is a solution of (60) for any integer l.
From (60) we find

$$\hat{Q}(w) = \frac{g\Omega_S^2}{U_C}\,\tilde{G}_N\,(r_+\omega_0 + w),\ 0 \le r_+ \le \frac{n - B(N)}{2} + B(N) \tag{67}$$

$$\hat{Q}(w) = \frac{g\Omega_S^2}{U_C}\,\tilde{G}_N^*\,(r_-\omega_0 - w^*),\ 0 \le r_- \le \frac{n - B(N)}{2} \tag{68}$$

where $B(N)$ is defined as

$$B(N) = \begin{cases} 0 & \text{for even } N \\ 1 & \text{for odd } N. \end{cases} \tag{69}$$

Equations (67)-(69) indicate that the minimum bandwidth needed for a multi-bunch feedback system is defined by $N\omega_0/2$. However, this is only possible if the upper and lower oscillation sidebands are treated the right way. In order to damp all modes, the phase (adjusted for damping) has to change sign at multiples of the revolution lines because (68) contains the complex conjugates of the transfer functions. These properties are well known from the theory of multi-bunch instabilities which is not surprising since "impedances" and transfer functions are equivalent quantities.

Fig. 1a illustrates the contributions from a periodic filter $\tilde{G}_N\,(\omega)$ for a fixed frequency $r\omega_0 + w$. The heavy lines describe an original analog filter $\tilde{G}\,(\omega)$ while $\tilde{G}_N$ is described by the thin lines.
At a fixed frequency ($r\omega_0 + w$ solid line, $r\omega_0$ dashed line) we find a direct contribution $\tilde{G}\,(r\omega_0 + \omega)$ and a contribution from a shifted part of $\tilde{G}$ (shaded area). This can be written as (see Fig. 1b)

$$\tilde{G}_N\,(r\omega_0 + \omega) = \tilde{G}\,(r\omega_0 + \omega) + \tilde{G}^*\,([N - r]\omega_0 - \omega), \tag{70}$$

which also corresponds to the impedance contributions in the theory of multi-bunch instabilities.

Approximations and estimates

Let us consider solutions of (67)-(69) for small g, i.e. $w \approx \Omega_S$. In this case we obtain using (54) for minimum bandwidth (see Fig. 1b)

$$(\Omega_S - w) \approx g\frac{\Omega_S}{2U_C}\,\tilde{G}\,(r_+\omega_0 + \Omega_S) \tag{71}$$

$$(\Omega_S - w) \approx g\frac{\Omega_S}{2U_C}\,\tilde{G}^*\,(r_-\omega_0 - \Omega_S) \tag{72}$$

where the functions $\tilde{G}$ have been used in the sense of eq. (70). Separating (71)-(72) with respect to amplitude and phase we arrive at

$$(\Omega_S - w) \approx g\frac{\Omega_S}{2U_C}|\,\tilde{G}\,|e^{i\varphi(r_+\omega_0 + \Omega_S)} \tag{73}$$

$$(\Omega_S - w) \approx g\frac{\Omega_S}{2U_C}|\,\tilde{G}\,|e^{-i\varphi(r_-\omega_0 - \Omega_S)} \tag{74}$$

Digital filters

As mentioned before, all mode-damping requires

$$\varphi(n\omega_0 + \Omega_S) = -\varphi(n\omega_0 - \Omega_S) \tag{75}$$

in the frequency range used for the feedback system. In [1] and [2] the single bunch related digital filter used for the PETRA feedback system has the effect of a phase shifter where the phase φ can be arbitrarily adjusted:

$$\tilde{P}(w) = e^{-iwT}\left\{\cos\varphi + \frac{4}{\pi} i \sin wT \sin\varphi\right\}. \tag{76}$$

This filter satisfies $\tilde{P}(w) = \tilde{P}(w + n\omega_0)$ for any integer n. Therefore carrying through the calculations (26)-(35) $\tilde{P}(\omega)$ will appear as a factor in eqs. (67)-(69), and the phase of $\tilde{P}(\omega)$ satisfies (75).

Estimated damping rates

From (71)-(72) we can estimate the damping rates expressed in terms of the inverse damping time τ_D

$$\frac{1}{\tau_D} \approx \frac{\Delta U_F/\Delta\Phi}{2U_C}\Omega_S \tag{77}$$

where $\Delta U_F/\Phi$ is the accelerating voltage per phase displacement $\Delta\Phi$.

B. Transverse Direction

In the transverse direction we start with the definition of the quasitime $\tau(t)$:

$$\tau(t) = \frac{\varphi(s[t])}{\omega_0 Q_\beta}, \; \omega_0 Q_\beta = \Omega_\beta \tag{78}$$

where $\varphi(s)$ is the phase advance along the orbit coordinate s which depends on t for a moving particle, Q_β is the transverse tune.
From eq. (78) we find immediately

$$\tau(t+T) = \tau(t) + T \tag{79}$$

and

$$\frac{d\tau}{dt} = \frac{1}{\eta}, \; \eta = \frac{\beta(s)}{R/Q_\beta}. \tag{80}$$

Here $\beta(s)$ is the amplitude function and R denotes the machine radius.

Signals

If X_ν is the transverse displacement of a single bunch, we introduce the Courant-Snyder coordinate according to

$$\xi_\nu = \frac{X_\nu}{\sqrt{\beta}}. \tag{81}$$

After that we define the bunch displacements in analogy to the longitudinal case as $\xi_\nu(\tau(t) - \tau(\nu T_B))$ and we have to investigate the transfer of signals like

$$S_\beta(t) = \bar{a}\sum_\nu \sum_{(k)} \xi_\nu(\tau(t) - \tau(\nu T_B)) T\delta(t - kT - \nu T_B) \cdot \sqrt{\beta_{pu}} \tag{82}$$

where β_{pu} is the amplitude function at the local pick-up station.

The deflecting forces follow from (82) as a linear transfer

$$F(t) = \bar{b} G_\beta[S_\beta(t)] \tag{83}$$

Equations of motion

The equations of motion can be written as

$$\frac{d^2}{d\tau^2}\xi_\mu(\tau - \tau(\mu T_B)) + \Omega_\beta^2 \xi(\tau - \tau(\mu T_B)) = \tag{84}$$
$$= \Omega_\beta^2 \beta_K^{3/2} \sum_{(k)} F(t) T\delta(t - kT - \mu T_B - t_B) + \sum_k g(t) T\delta(t - kT - \mu T_B)$$

As in the longitudinal case $g(t)$ describes an external test excitation, while β_K is the amplitude function at the kicker, and t_B denotes the distance between pick-up and kicker measured in time units.

Signal transfer

Because of (79) we can rewrite the signal (82) as

$$S_\beta(t) = \bar{a}\sum_\nu \sum_{(k)} \xi_\nu(kT) T\delta(t - kT - \nu T_B)\sqrt{\beta_{pu}}. \tag{85}$$

$$S_\beta(t) = \bar{a}\sum_\nu \sum_{(k)} \xi_\nu(t - \nu T_B) T\delta(t - kT - \nu T_B)\sqrt{\beta_{pu}}. \tag{86}$$

The generalized Fourier representation of ξ is introduced by

$$\xi(\tau) = \int_L dw' \,\tilde{\xi}(w') e^{iw'\tau}. \tag{87}$$

In analogy to equations (37)-(38) we make the replacement $\tau_\mu = \tau - \tau(\mu T_B)$ on the l.h.s. of (84), these equations read

$$\frac{d^2}{d\tau_\mu^2}\xi_\mu(\tau_\mu) + \Omega_\beta^2 \xi_\mu(\tau_\mu) = \Omega_\beta^2 \beta_K^{3/2} \sum_{(k)} F(t) T\delta(t - kT - \mu T_B - t_B) + \tag{88}$$

$$+\sum_{(k)} g(t)T\delta(t-kT-\mu T_B)$$

We introduce the abbreviations

$$F_\mu(t)=\sum_{(k)} F(t)\delta(t-kT-\mu T_B-t_B)T \tag{89}$$

$$g_\mu(t)=\sum_{(k)} g(t)T\delta(t-kT-\mu T_B) \tag{90}$$

and proceed as in the longitudinal case. Therefore we have to evaluate

$$\tilde{F}_\mu(w)=\frac{1}{2\pi}\int d\tau_\mu F_\mu(t)e^{-iw\tau_\mu}=\frac{1}{2\pi}\int d\tau F_\mu(t)e^{-iw(\tau(t)-\tau(\mu T_B))} \tag{91}$$

The integration over τ can be replaced by an integration over t with the help of the relation (80)

$$\tilde{F}_\mu(w)=\frac{R}{Q_\beta}\frac{1}{\beta_K}\frac{1}{2\pi}T\sum_{(k)} F(kT+\mu T_B+t_B)e^{-ikwT}e^{-iw\tau_B} \tag{92}$$

where τ_B is defined as $\tau_B=\tau(t_B)$.
Now we have to express $F(kT+\mu T_B+t_B)$ as a linear functional of $S_\beta(t)$ according to (83), and this can be performed in analogy to the steps between equation (29) and (34).

There is a special difference from the longitudinal case because (92) explicitly contains t_B resp. τ_B.

In the longitudinal case, the phase advance between the pick-up and the cavities has been dropped because the synchrotron period is very small compared to the revolution time. But this is completely different in the transverse case.
Following the procedure outlined in the longitudinal case (eqs. (32)-(34)), we obtain explicitly

$$F(kT+\mu T_B+t_B)=T\bar{b}\bar{a}\sum_\tau\int_L dw'\frac{1}{\sqrt{N}}\tilde{C}_\tau^{(\beta)}(w')\,\tilde{G}_{CN}^{(\beta)}(w'+r\omega_0)e^{i(w'+r\omega_0)kT}e^{2\pi i\mu\tau/N}e^{iw'\mu T_B} \tag{93}$$

with

$$\tilde{G}_{CN}^{(\beta)}(w'+r\omega_0)=\sum_{(l)}\tilde{G}_C^{(\beta)}(w'+r\omega_0+lN\omega_0) \tag{94}$$

and

$$\tilde{G}_C^{(\beta)}(w'+r\omega_0+lN\omega_0)=\tilde{G}^{(\beta)}(w'+r\omega_0+lN\omega_0)e^{i(w'+r\omega_0+lN\omega_0)t_B} \tag{95}$$

Let us consider the t_B-dependence in (94).
In order to arrange the coincidence of signal and bunch arrival at the kicker (see Fig. 2) we have to provide a proper adjustable delay τ_F.
The transfer function $\tilde{G}^{(\beta)}$ of the feedback system contains two parts, the transfer function of all analog components $G_a^{(\beta)}$ (amplifiers, phase detectors, kickers, etc.) and the adjustable filters which provide damping $\tilde{D}^{(\beta)}$.

$$\tilde{G}^{(\beta)}(w'+r\omega_0+lN\omega_0)= \tag{96}$$

$$=\tilde{G}_a^{(\beta)}(w'+r\omega_0+lN\omega_0)|\,\tilde{D}^{(\beta)}\,|e^{i\varphi(w'+r\omega_0+lN\omega_0)}e^{-i\tau_F(w'+r\omega_0+lN\omega_0)}.$$

Here φ denotes the adjustable phase (phase shifter) and τ_F is an adjustable delay. Now we rewrite $\tilde{G}_a^{(\beta)}$ separating amplitude and phase

$$\tilde{G}_a^{(\beta)}=|\,\tilde{G}_a^{(\beta)}(w'+r\omega_0+lN\omega_0)|e^{i\chi(w'+r\omega_0+lN\omega_0)} \tag{97}$$

The phase χ can be decomposed according to

$$\chi(v)=\varphi_0(v)-t_E\cdot v+\Delta\chi(v). \tag{98}$$

This, of course, is not unique, however, φ_0 and t_E can be chosen in such a way that $\Delta\chi(v)$ has minimum values for all frequencies v within the frequency range of the feedback system. From Fig. 2 we find for the coincidence of signal and bunch arrival at the kicker

$$t_B+nT=t_E+\tau_F \tag{99}$$

or

$$t_B-t_E-\tau_F=-nT \tag{100}$$

where n is an appropriate positive integer number.
From (94), (96)-(99) then follows

$$\tilde{G}^{(\beta)}e^{i(w'+r\omega_0+lN\omega_0)t_B}=\tilde{G}_C^{(\beta)}(w'+r\omega_0+lN\omega_0)= \tag{101}$$

$$=|\,\tilde{G}_a^{(\beta)}\,||\,\tilde{D}^{(\beta)}\,|e^{i(\varphi+\varphi_0)}e^{-inw'T}e^{i\Delta\chi(w'+r\omega_0+lN\omega_0)},$$

so that the t_B dependence disappears in $F(kT+\mu T_B+t_B)$ due to the coincidence conditions. The two adjustable parameters φ, τ_F then provide the optimization of the damping rate for all frequencies (modes). This, of course, is only possible if all phase deviations are small compared to $\pi/2$.

Coupled bunch equations

Now we proceed in analogy to the steps between eqs. (43) and (51) and arrive at

$$\hat{Q}_B^{(\beta)}(w)\hat{\xi}_\mu(w)=\bar{g}\frac{R}{Q_\beta}\Omega_\beta^2\cdot\sqrt{\beta_{p\mu}}\sqrt{\beta_K}\sum_\nu\hat{G}_{\mu\nu}^{(\beta)}(w)\hat{\xi}_\nu(w)+\hat{g}_\mu(w) \tag{102}$$

with

$$\tilde{R}_B^{(\beta)}(w)=\sum_{(k)}\frac{e^{-i(w+k\omega_0)\tau_B}}{\Omega_\beta^2-(w+k\omega_0)^2}=1/\hat{Q}_B^{(\beta)}(w) \tag{103}$$

and

$$\hat{G}_{\mu\nu}^{(\beta)}(w)=\sum_{(l)}\frac{1}{\sqrt{N}}e^{i(w+l\omega_0)\mu T_B}\,\tilde{G}_C^{(\beta)}(w+l\omega_0)\frac{1}{\sqrt{N}}e^{-i(w+l\omega_0)\nu T_B}. \tag{104}$$

In order to exhibit the special structure of betatron oscillations we introduce the "fractional" betatron frequencies

$$\begin{aligned}\omega_\beta&=\Omega_\beta-q_\beta\cdot\omega_0\\ u&=w-q_\beta\omega_0\end{aligned} \tag{105}$$

where q_β is the integer part of Q_β. Obviously this yields for $G^{(\beta)}_{\mu\nu}$ and $\hat{R}^{(\beta)}_B$

$$\hat{G}^{(\beta)}_{\mu\nu}(w) = \hat{G}^{(\beta)}_{\mu\nu}(u);\; \hat{R}^{(\beta)}_B(w) = \hat{R}^{(\beta)}_B(u). \tag{106}$$

The function $\hat{R}^{(\beta)}_B$ has been evaluated in A2 and one finds

$$\hat{Q}^{(\beta)}_B(u) \overset{u\to\omega_\beta}{\longrightarrow} 2\Omega_\beta(\omega_\beta - u)e^{i\varphi_{pk}},\; \varphi_{pk} = \Omega_\beta \tau_B. \tag{107}$$

where φ_{pk} is the phase advance between the pick-up and the kicker.

Eigenfrequencies of the system

In analogy to the arguments leading to eqs. (59)-(60) we arrive at

$$\hat{Q}^{(\beta)}_B(u) = \overline{g}\frac{R}{Q_\beta}\Omega^2_\beta\sqrt{\beta_{pu}}\sqrt{\beta_k}\,\tilde{G}{}^{(\beta)}_{CN}(r\omega_0 + u) \tag{108}$$

and in analogy to eqs. (71)-(72) for minimum bandwidth these equations reduce to

$$(\omega_\beta - u) \approx \overline{g}\frac{R}{Q_\beta}\frac{\Omega_\beta}{2}\sqrt{\beta_{pu}}\sqrt{\beta_k}e^{-i\varphi_{pk}}\,\tilde{G}{}^{(\beta)}_C(r_+\omega_0 + \omega_\beta) \tag{109}$$

$$(\omega_\beta - u) \approx \overline{g}\frac{R}{Q_\beta}\frac{\Omega_\beta}{2}\sqrt{\beta_{pu}}\sqrt{\beta_k}e^{-i\varphi_{pk}}\,\tilde{G}{}^{(\beta)*}_C(r_-\omega_0 - \omega_\beta). \tag{110}$$

According to the physical meaning of $F(t)$ in (87), the constant $\overline{g}$ can be expressed as

$$\overline{g} = \frac{\Delta x'}{\Delta x}/2\pi R \tag{111}$$

where $\Delta x'/\Delta x$ denotes the available kick per displacement. Finally we get ($\omega_0 = 2\pi f_0$)

$$(\omega_\beta - u) \approx \frac{\Delta x'}{\Delta x}\sqrt{\beta_{pu}}\sqrt{\beta_k}\frac{f_0}{2}\,\tilde{G}{}^{(\beta)}_C(r_+\omega_0 + \omega_\beta)e^{-i\varphi_{pk}} \tag{112}$$

$$(\omega_\beta - u) \approx \frac{\Delta x'}{\Delta x}\sqrt{\beta_{pu}}\sqrt{\beta_k}\frac{f_0}{2}\,\tilde{G}{}^{(\beta)}_C(r_-\omega_0 + \omega_\beta)e^{-i\varphi_{pk}} \tag{113}$$

which leads to a rough estimate of the damping rate $1/\tau_{D\beta}$

$$\frac{1}{\tau_{D\beta}} \approx \frac{\Delta x'}{\Delta x}\sqrt{\beta_{pu}}\sqrt{\beta_k}\frac{f_0}{2}. \tag{114}$$

5. Single Bunch Motion

Motion around the ring

In this section we consider the explicit time dependence of the displacement coordinate of a single bunch. For simplicity, we restrict ourselves to the longitudinal direction. From eqs. (49) and (56) we obtain

$$\tilde{\Phi}_\mu(w) = \frac{1}{\Omega^2_S - w^2}\gamma\sum_r\left\{G_r(w)\hat{C}_r(w)\overline{K}_{r\mu}(w) + \frac{1}{\Omega^2_S - w^2}F_r(w)\overline{K}_{r\mu}(w)\right\} \tag{115}$$

with the abbreviations

$$\gamma = g\frac{\Omega^2_S}{U_C};\; G_r(w) = \tilde{G}_N(r\omega_0 + w). \tag{116}$$

Using (59) we obtain

$$\tilde{\Phi}_\mu(w) = \frac{1}{\Omega^2_S - w^2}\sum_r\left\{\frac{\gamma G_r(w)F_r(w)}{\hat{Q}(w) - \gamma G_r(w)} - F_r(w)\right\}\overline{K}_{r\mu}(w) \tag{117}$$

$$\tilde{\Phi}_\mu(w) = \frac{1}{(\Omega^2_S - w^2)\hat{R}(w)}\sum_r\frac{F_r(w)}{\hat{Q}(w) - \gamma G_r(w)}\overline{K}_{r\mu}(w).$$

According to eq. (58), $F_r(w)$ can be expressed by

$$F_r(w) = \sum_\nu \overline{K}^*_\mu(w)\hat{\varphi}_\nu(w) \tag{118}$$

so that we finally arrive at

$$\tilde{\Phi}_\mu(w) = \frac{1}{(\Omega^2_S - w^2)\hat{R}(w)}\sum_\nu \hat{P}_{\mu\nu}(w)\hat{\varphi}_\nu(w), \tag{119}$$

where the resolvent $\hat{P}_{\mu\nu}(w)$ is defined by its spectral representation

$$\hat{P}_{\mu\nu}(w) = \sum_r \overline{K}_{r\mu}(w)\frac{1}{\hat{Q}(w) - \gamma G_r(w)}\overline{K}^*_{r\nu}(w). \tag{120}$$

As in the case of $\hat{G}_{\mu\nu}(w)$, the formal considerations of A1 lead to the periodicity property

$$\hat{P}_\mu(w) = \hat{P}_\mu(w + l\omega_0) \text{ for all integers } l \tag{121}$$

Therefore, if w_r is a pole of $\hat{P}_\mu$, then $w_r + l\omega_0$ is also a pole of $P_{\mu\nu}$ in agreement with the result of eqs. (64)-(66).

Because of (121), the residua of $\hat{P}_\mu(w)$ at the poles $w_r + l\omega_0$ are equal for all integer l, and we set

$$\text{Res}\;\hat{P}_{\mu\nu}(w_r + l\omega_0) = a_{\mu\nu}(w_r). \tag{122}$$

In order to simplify the discussion on the time dependences of the phase deviations, we replace $\varphi_\nu(t)$ by one-turn kicks. In that case, the $\hat{\varphi}_\nu(w)$ reduce to constants

$$\hat{\varphi}_\nu(w) \to k_\nu, \tag{123}$$

and the time dependence of the phase deviations is only determined by the poles of $\hat{P}_{\mu\nu}$. Using eq. (4) and the residuum theorem we obtain

$$\Phi_\mu(t) = 2\pi i \sum_r \sum_{(l)} \sum_\nu \frac{a_{\mu\nu}(w_r)k_\nu}{[\Omega_S^2 - (w + l\omega_0)^2]\hat{R}(w_r)} e^{i(w_r + l\omega_0)t}, \qquad (124)$$

and the time dependence is essentially determined by the sum

$$S(t,w) = \sum_{(l)} \frac{e^{i(w_r + l\omega_0)t}}{\Omega_S^2 - (w_r + l\omega_0)^2}; \quad t \geq 0 \qquad (125)$$

which is evaluated in the Appendix A2. The result reads

$$S(w_r, t) = -\frac{i\pi}{\Omega_S \omega_0} \left\{ \frac{e^{i\Omega_S t}(ax_r)^{-n}}{1 - ax_r} - \frac{e^{-i\Omega_S t}(a^* x_r)^{-n}}{1 - a^* x_r} \right\} \qquad (126)$$

for $\quad nT \leq t < (n+1)T$,
with $\quad x = e^{-iw_r T}$
where n is a positive integer number.
It follows from eq. (126) that between the cavity passages $\Phi_\mu(t)$ describes a free oscillation with the unperturbed frequency Ω_S. After a cavity passage the amplitude is modified by the factors $1/ax_r$ resp. $1/a^* x_r$. In the case of a pure damping $w = \Omega_S + i\Gamma_r$ these factors reduce to a damping factor $e^{-\Gamma_r T}$, an expected result.

Displacements at a local pick-up

From (119)-(120) we immediately obtain

$$\hat{\Phi}_\mu(w) = \sum \hat{P}_{\mu\nu}(w)\hat{\varphi}_\nu(w) \qquad (127)$$

and from the definition of $\hat{\Phi}_\mu(w)$ we find the relation

$$\hat{\Phi}_\mu(t) = \int dw \hat{\Phi}_\mu(w) e^{iwt} = \sum_k \Phi_\mu(kT)\delta(t - kT)T. \qquad (128)$$

If w_r^0, $r = 0, \ldots N-1$ are poles with the lowest real frequencies, we can rewrite $\hat{\Phi}_\mu(t)$ as

$$\hat{\Phi}_\mu(t) = 2\pi i \sum_r \sum_\nu a_{\mu\nu}(w_r)k_\nu \sum_{(l)} e^{i(w_r + l\omega_0)t}. \qquad (129)$$

Because of

$$\sum_{(l)} e^{il\omega_0 t} = T \sum_{(k)} \delta(t - kT)$$

(129) becomes

$$\hat{\Phi}_\mu(t) = 2\pi i \sum_r \sum_\nu a_{\mu\nu}(w_r)k_\nu e^{iw_r t} \sum_{(k)} T\delta(t - kT). \qquad (130)$$

Comparing (130) with (128) yields

$$\hat{\Phi}_\mu(kT) = 2\pi i \sum_r \sum_\nu a_{\mu\nu}(w_r)k_\nu e^{iw_r kT}, \qquad (131)$$

again an expected result.

Acknowledgements

The author is indebted to Dr. K. Balewski for numerous intensive discussions.
The author is grateful to S.G. Wipf for carefully reading the manuscript.

References

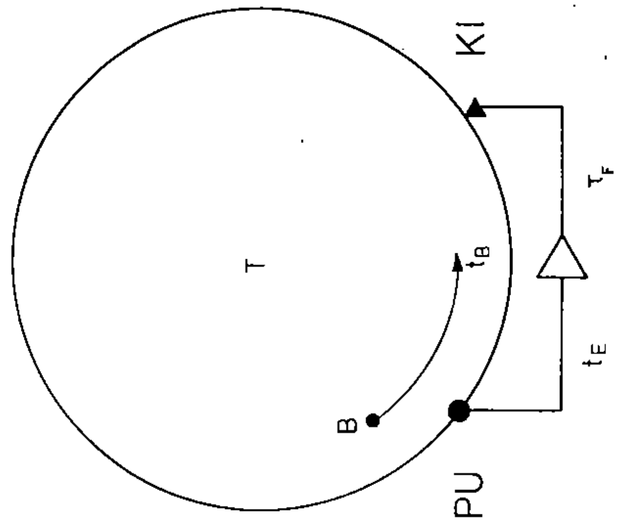
[1] D. Heins et al. DESY 89-157

[2] M. Ebert et al. DESY 91-036

Figure Captions

Fig. 1 Illustration of periodic filters

Fig. 2 Delay conditions for the feedback system



$t_E > t_B$

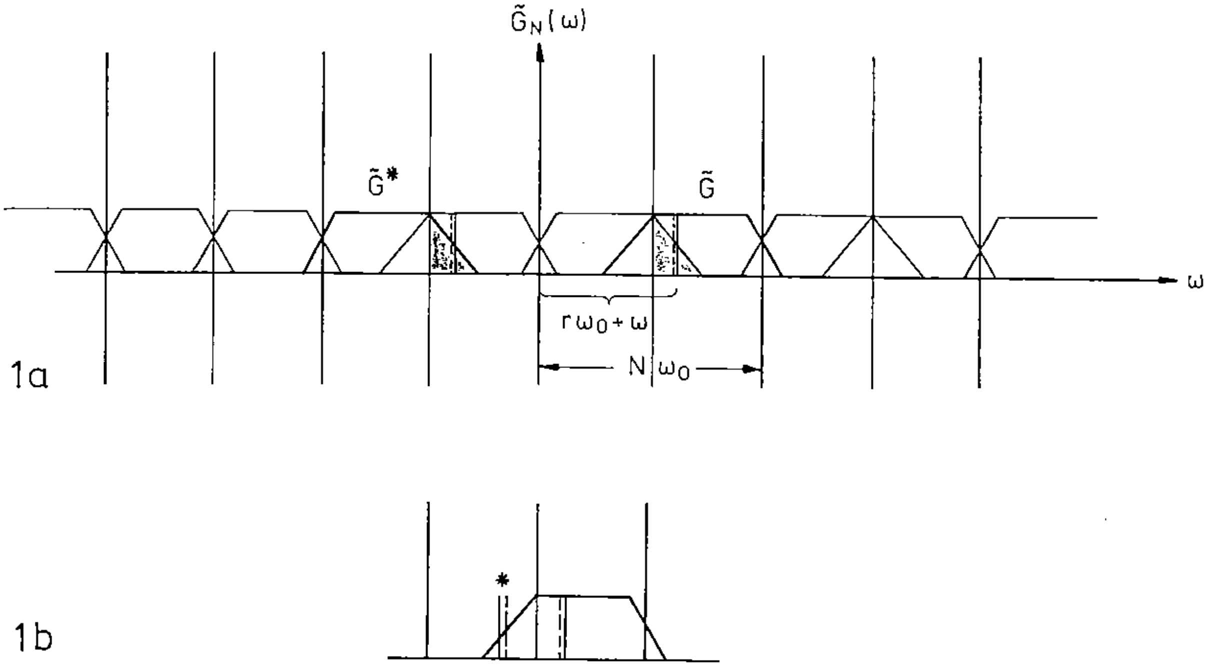
condition for coincidence of signal and bunch arrival

$$t_B + m \cdot T = t_E + \tau_F$$

$$\tau_F = m\,T - (t_E - t_B)$$

$$\mathrm{Min}(\tau) \geq 0$$

Fig. 2 delay conditions for the feedback system



$$\tilde{G}_N(r\omega_0 + \omega) = \tilde{G}(r\omega_0 + \omega) + \tilde{G}(-[N\omega_0 - r\omega_0 - \omega])$$
$$= \tilde{G}(r\omega_0 + \omega) + \tilde{G}^*((N-r)\,\omega_0 - \omega)$$

Fig. 1 Illustration of periodic filters

A APPENDIX 1

We consider a sum like (46)

$$\sigma(w) = \sum_{(k)} g(w + k\omega_0). \tag{A11}$$

This sum can be replaced by

$$\sigma(w) = \sum_m \sum_{(l)} g(w + lN\omega_0 + m\omega_0), \; m = 0, \ldots, N-1 \tag{A12}$$

which is evident since any integer k can be represented by

$$k = lN + m, \; 0 \leq m \leq N-1.$$

The equation for the voltage (44) has the general form

$$Y_\mu(w) = \sum_r \sum_{(k)} \sum_\nu A_\nu(w - r\omega_0 + k\omega_0) B_{N\mu\nu}(w + k\omega_0) \tag{A13}$$

where $B_{N\mu\nu}(w')$ has the property

$$B_{N\mu\nu}(w') = B_{N\mu\nu}(w' + lN\omega_0) \tag{A14}$$

for all integers l. We substitute

$$k - r = p \tag{A15}$$

and obtain

$$Y_\mu(w) = \sum_r \sum_{(p)} \sum_\nu A_\nu(w + p\omega_0)\, B_{N\mu\nu}(w + r\omega_0 + p\omega_0). \tag{A16}$$

Applying A12 yields because of A14

$$Y_\mu(w) = \sum_r \sum_m \sum_{(q)} \sum_\nu A_\nu(w + m\omega_0 + qN\omega_0)\, B_{N\mu\nu}(w + r\omega_0 + m\omega_0) \tag{A17}$$

or introducing

$$\sum_{(q)} A_\nu(w + m\omega_0 + qN\omega_0) = A_{N\nu}(w + m\omega_0) \tag{A18}$$

$$Y_\mu(w) = \sum_r \sum_m \sum_\nu A_{N\nu}(w + m\omega_0) B_{N\mu\nu}(w + r\omega_0 + m\omega_0).$$

Because of A14 we can add an arbitrary integer multiple of $N\omega_0$ in the argument of $B_{N\mu\nu}$ in A18. We define an integer

$$l(m, r) = \begin{cases} 0 \text{ for } r + m \leq N \\ -1 \text{ for } r + m > N \end{cases} \tag{A19}$$

so that $m + r + l(m,r)N$ runs through all integer values between $0 \ldots N-1$ if r runs from 0 to $N-1$. Therefore we obtain from A18 by applying A12

$$\sum_r B_{N\mu\nu}(w + [m + r + lN\omega_0]) = \sum_{(k)} B_{\mu\nu}(w + k\omega_0) = \hat{B}_{\mu\nu}(w) \tag{A20}$$

Applying A12 again the remaining sum over m leads to the relation

$$Y_\mu(w) = \sum_\nu \hat{A}_\nu(w) \hat{B}_{\mu\nu}(w) \tag{A21}$$

used for the deviation of eq. (45).

B APPENDIX 2

In this appendix we will derive an explicit expression for

$$S_\Omega(t, w) = \sum_{(l)} \frac{e^{i(w + l\omega_0)t}}{\Omega^2 - (w + l\omega_0)^2}. \tag{A22}$$

Except for poles on the real axis $S_\Omega(t, w)$ is an analytic function of w. Therefore if we know S_Ω for the lower half plane, the sum A22 can be replaced by an integral using the periodic δ-function

$$S_\Omega(t, w) = \int d\omega \delta_p(w) \frac{e^{i(w+\omega)t}}{\Omega^2 - (w + \omega)^2} \tag{A23}$$

$$\delta_p(w) = \frac{1}{2\pi} \int d\zeta e^{iw\zeta} T \sum_{(k)} \delta(\zeta - kT) \tag{A24}$$

We decompose the denominator in A23 according to

$$-\frac{1}{(w + \omega)^2 - \Omega^2} = -\frac{1}{2\Omega}\left(\frac{1}{w + \omega - \Omega} - \frac{1}{w + \omega + \Omega}\right). \tag{A25}$$

From this equation the poles for ω follow immediately

$$\omega_1 = \Omega - w \text{ and } \omega_2 = -\Omega - w \tag{A26}$$

these are located in the upper ω plane because of $Im w < 0$, and we can derive the following relation

$$V(t, \zeta) = \frac{1}{2\pi} \int d\omega \frac{e^{i(w+\omega)t} e^{i\omega\zeta}}{\Omega^2 - (w + \omega)^2} = -\frac{i}{2\Omega} \Theta(t + \zeta) \left\{ e^{i\Omega t} e^{i(\Omega - w)\zeta} - e^{-i\Omega t} e^{-i(\Omega + w)\zeta} \right\}. \tag{A27}$$

According to A24 and A26 we find for $S_\Omega(t, w)$

$$S_\Omega(t, w) = \sum_{(k)} V_\Omega(t, kT) \tag{A28}$$

1. t / 0

In that case the sum A28 is extended only over $k \geq 0$ because of the Θ-function in A26. The result can be written as

$$S_\Omega(0, w) = -\frac{i\pi}{\Omega\omega_0} \left\{ \frac{1}{1 - ax} - \frac{1}{1 - a^* x} \right\}, \tag{A29}$$

with $x = e^{-iwT}$ and $a = e^{i\Omega T}$,

2. $t = -\tau;\; \tau > 0$

In that case the sum A27 is extended only over $k \geq 1$ and the result is

$$S_\Omega(-\tau, w) = -i \frac{\pi}{\Omega\omega_0} \left\{ \frac{e^{-i\Omega\tau} ax}{1 - ax} - \frac{e^{+i\Omega\tau} a^* x}{1 - a^* x} \right\}. \tag{A30}$$

The expressions A29 and A30 can be analytically continued for arbitrary complex w-values except for the poles.
For $w \to \Omega$ we obtain from A29 and A30

$$1/S_\Omega(0,w) \to 2\Omega(\Omega - w) \tag{A31}$$

$$1/S_\Omega(-\tau,w) \to 2\Omega(\Omega - w)e^{+i\Omega\tau} \tag{A32}$$

3. $0 \le t < T$

The sum A28 is extended over $k \ge 0$. This leads to

$$S_\Omega(t,w) = -\frac{i\pi}{\Omega\omega_0}\left\{\frac{e^{i\Omega t}}{1-ax} - \frac{e^{-i\Omega t}}{1-a^*x}\right\} \tag{A33}$$

4. $T \le t < 2T$

In that case the sum A28 is extended over $k \ge 0$ including also $k = -1$.
We find

$$S_\Omega(t,w) = -i\frac{\pi}{\Omega\omega_0}\left\{\frac{e^{i\Omega t}/ax}{1-ax} - \frac{e^{-i\Omega t}/a^*x}{1-a^*x}\right\}. \tag{A34}$$

And finally we arrive at

$$S_\Omega(t,w) = -i\frac{\pi}{\Omega\omega_0}\left\{\frac{e^{i\Omega t}/(ax)^{-n}}{1-ax} - \frac{e^{-i\Omega t}/(a^*x)^{-n}}{1-a^*x}\right\} \tag{A35}$$

with $nT \le t < (n+1)T$
for positive integer n.